

# When is a cleft extension $H$ -Azumaya?

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## Abstract

We show which  $H^{op}$ -cleft extensions  $\mathbf{k} \#_{\sigma} H^{op}$  of  $\mathbf{k}$  for a dual quasitriangular Hopf algebra  $(H, r)$  are  $H$ -Azumaya. The result is given in terms of bijectivity of a map  $\theta_{\sigma}: H \rightarrow H^*$  defined in terms of  $r$  and the 2-cocycle  $\sigma$ , generalizing a well-known result for the commutative and cocommutative case. We illustrate the Theorem with an explicit computation for the Hopf algebras of type  $E(n)$ .

**Key words:** Cleft extension, dual quasitriangular Hopf algebra,  $H$ -Azumaya algebra

**MSC:**16W30, 16H05, 16K50

## Introduction

In the last fifty years several generalizations of the classical Brauer group of a field  $\mathbf{k}$  were introduced. A key role in algebraic geometry is played by the Brauer group of suitable sheaves of algebras over a scheme  $X$  ([11]). In representation theory and in ring theory generalizations are obtained either relaxing the requirements on  $\mathbf{k}$  and/or on the algebras, or by adding extra structures. In the first case one can, for instance, allow  $\mathbf{k}$  to be a commutative ring ([1]) or allow algebras without unit ([20]). In the second case one can work in categories whose objects carry more structure, such as a grading by an abelian group ([24], [13]), a commutative and cocommutative Hopf algebra action ([14]), etcetera. Most of the known generalizations are examples of the Brauer group of a symmetric monoidal category ([17]). The Brauer group of a general Hopf algebra ([5]) and later, the Brauer group

of a braided monoidal category ([21]) seem to be the highest level of generality for a Brauer group in this setting that has been reached so far. The most popular examples of a braided monoidal category are the category of left modules over a quasitriangular Hopf algebra and, dually, the category of right comodules of a dual quasitriangular Hopf algebra. In these cases the elements of the corresponding Brauer group are equivalence classes of particular module and comodule algebras, respectively. The explicit computation of the corresponding Brauer group is in general far from being trivial. When studying these groups it is quite natural to wonder whether well-known modules and comodules algebras can be seen as representatives of elements of these Brauer groups, that is, whether they are Azumaya algebras in the corresponding category. A very well-known family of comodule algebras is provided by cleft extensions of the base field ([10]).

Cleft extensions are a natural generalization of Galois extensions. In the finite-dimensional case the notion of cleft extension coincides with the notion of a Hopf-Galois extension of a Hopf algebra. In the general case the notion of cleft extension is stronger than the notion of a Hopf-Galois extension because the existence of a normal basis is required. The theory of cleft extensions has reached a satisfactory description in terms of crossed systems (see [10], [3]). Cleft extensions of the base field correspond to equivalence classes of 2-cocycles and they are isomorphic, as algebras, to twists of  $H$  by the corresponding cocycle. A Theorem in [4, §12.4] expresses explicitly when the twist of a commutative, cocommutative Hopf algebra by a Sweedler cocycle is  $H$ -Azumaya. The aim of the present paper is the generalization of this theorem to the dual quasitriangular case and this is reached in Theorem 2.1. In analogy to the cocommutative case the result states that such a twist is Azumaya in the category if and only if the linear map  $\theta_\sigma: H \longrightarrow H^*$  defined by  $h \mapsto (\sigma\tau * r * \sigma^{-1})(- \otimes h)$  is invertible. After the proof of Theorem 2.1 we discuss a dual version of the result obtaining Corollary 2.6. The main result is illustrated with an example: for every universal  $r$ -form  $r$  of the Hopf algebras of type  $E(n)$  introduced in [2] we characterize in Proposition 3.1 which twists of  $E(n)$  are Azumaya with respect to  $r$ . Finally, in Proposition 3.3, we relate this description to the computation of the Brauer group  $BM(\mathbf{k}, E(n), R_0)$  of the category of modules of  $E(n)$  with respect to the  $R$ -matrix  $R_0$  obtained in [9].

# 1 Cleft extensions and $H$ -Azumaya algebras

Unless otherwise stated  $H$  will denote a finite-dimensional Hopf algebra over a field  $\mathbf{k}$ , with coproduct  $\Delta$  and antipode  $S$ . All modules, comodules and algebras will be assumed to be over  $\mathbf{k}$ , as well as unadorned tensor products. The standard flip map  $V \otimes W \rightarrow W \otimes V$  will be denoted by  $\tau$ . For coproduct and right comodule structures we shall use the notations  $\sum n_{(1)} \otimes n_{(2)}$  and  $\sum n_{(0)} \otimes n_{(1)}$ , respectively.

The Brauer group of a braided monoidal category was defined in [21]. It is well-known that if  $H$  is a dual quasitriangular Hopf algebra with universal  $r$ -form  $r$  the category  $\mathcal{M}^H$  of finite-dimensional right  $H$ -comodules is braided monoidal with  $H$ -comodule structure on the tensor product given by

$$\rho(m \otimes n) = \sum m_{(0)} \otimes n_{(0)} \otimes n_{(1)} m_{(1)}$$

and braiding  $\psi$  given by  $\psi(m \otimes n) = \sum n_{(0)} \otimes m_{(0)} r(n_{(1)} \otimes m_{(1)})$  for every pair of comodules  $M$  and  $N$  and every  $m \in M$  and  $n \in N$ . In this particular setting, an algebra in the category  $\mathcal{M}^H$  is an  $H^{op}$ -comodule algebra  $A$ . In particular, if  $P$  is a  $H$ -comodule then  $End(P)$  with the usual composition of endomorphisms and with comodule structure given by

$$\rho(f)(a) = \sum f(a_{(0)})_{(0)} \otimes S^{-1}(a_{(1)}) f(a_{(0)})_{(1)} \quad (1.1)$$

for every  $f \in End(P)$  and every  $a \in P$  is an algebra in  $\mathcal{M}^H$ . Similarly,  $End(P)^{op}$  is an  $H^{op}$ -comodule algebra with respect to the structure:

$$\rho(f)(a) = \sum f(a_{(0)})_{(0)} \otimes f(a_{(0)})_{(1)} S(a_{(1)}). \quad (1.2)$$

The opposite algebra  $\overline{A}$  of an algebra  $A$  in the category is equal to  $A$  as a  $H$ -comodule but its product is given by  $a \circ b = \sum b_{(0)} a_{(0)} r(b_{(1)} \otimes a_{(1)})$ . It is again an  $H^{op}$ -comodule algebra. Given two algebras  $A$  and  $B$  in the category we endow the comodule  $A \otimes B$  with the product

$$(a \# b)(c \# d) = \sum (ac_{(0)} \# b_{(0)} d) r(c_{(1)} \otimes b_{(1)})$$

for every  $a, c \in A$  and every  $b, d \in B$ . The resulting algebra is a  $H^{op}$ -comodule algebra, denoted by  $A \# B$ . An algebra  $A$  in  $\mathcal{M}^H$  is called Azumaya, or  $(H, r)$ -Azumaya if the  $H^{op}$ -comodule algebra maps

$$\begin{aligned} F: A \# \overline{A} &\longrightarrow End(A) \\ F(a \# \overline{b})(c) &= \sum ac_{(0)} b_{(0)} r(c_{(1)} \otimes b_{(1)}) \end{aligned}$$

and

$$\begin{aligned} G: \bar{A} \# A &\longrightarrow \text{End}(A)^{op} \\ G(\bar{a} \# b)(c) &= \sum r(a_{(1)} \otimes c_{(1)}) a_{(0)} c_{(0)} b \end{aligned}$$

are isomorphisms. The opposite algebra of an  $(H, r)$ -Azumaya algebra and the product  $\#$  of two  $(H, r)$ -Azumaya algebras are again  $(H, r)$ -Azumaya algebras. The elements of the Brauer group  $BC(\mathbf{k}, H, r)$  of the category  $\mathcal{M}^H$  are the equivalence classes of  $(H, r)$ -Azumaya algebras with respect to the equivalence relation:  $A \sim B$  if  $A \# \text{End}(P) \cong B \# \text{End}(Q)$  for some  $H$ -comodules  $P$  and  $Q$ . The product  $\#$  induces on  $BC(\mathbf{k}, H, r)$  a group structure and the inverse of a class represented by an algebra  $A$  is the class represented by the opposite algebra  $\bar{A}$ .

Dually, if  $H$  is a quasitriangular Hopf algebra with  $R$ -matrix  $R = \sum R^{(1)} \otimes R^{(2)}$ , the category  ${}_H\mathcal{M}$  of finite-dimensional left  $H$ -modules is monoidal, with usual  $H$ -module structure on the tensor product of two modules. An algebra in the category  ${}_H\mathcal{M}$  is just an  $H$ -module algebra. If  $P$  is a  $H$ -module then  $\text{End}(P)$  with the usual composition of endomorphisms and with module structure given by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m) \quad (1.3)$$

for every  $h \in H$ , every  $f \in \text{End}(P)$  and every  $m \in P$  is an algebra in  ${}_H\mathcal{M}$ . Similarly,  $\text{End}(P)^{op}$  is also an algebra in  ${}_H\mathcal{M}$  if we endow it with the module structure:

$$(h \cdot f)(m) = \sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m). \quad (1.4)$$

The opposite algebra  $\bar{A}$  of an algebra  $A$  in  ${}_H\mathcal{M}$  is equal to  $A$  as an  $H$ -module and its product is given by  $a \circ b = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a)$ . It is again an  $H$ -module algebra. The tensor product of two algebras  $A$  and  $B$  in the category  ${}_H\mathcal{M}$  is  $A \# B \cong A \otimes B$  as modules with the multiplication:

$$(a \# b)(c \# d) = \sum a(R^{(2)} \cdot c) \# (R^{(1)} \cdot b) \cdot d$$

for every  $a, c \in A$  and every  $b, d \in B$ . An algebra  $A$  in  ${}_H\mathcal{M}$  is called Azumaya, or  $(H, R)$ -Azumaya if the  $H$ -module algebra maps

$$\begin{aligned} F': A \# \bar{A} &\longrightarrow \text{End}(A) \\ F'(a \# \bar{b})(c) &= \sum a(R^{(2)} \cdot c)(R^{(1)} \cdot b) \end{aligned}$$

and

$$\begin{aligned} G': \bar{A} \# A &\longrightarrow \text{End}(A)^{op} \\ G'(\bar{a} \# b)(c) &= \sum (R^{(2)} \cdot a)(R^{(1)} \cdot c) b \end{aligned}$$

are isomorphisms. The elements of the Brauer group  $BM(\mathbf{k}, H, R)$  of the category  ${}_H\mathcal{M}$  are the equivalence classes of  $(H, R)$ -Azumaya algebras with respect to the equivalence relation:  $A \sim B$  if  $A \# \text{End}(P) \cong B \# \text{End}(Q)$  for some  $H$ -modules  $P$  and  $Q$ . The product in  $BM(\mathbf{k}, H, R)$  is induced by the product  $\#$ , with inverse represented by the opposite algebra.

Computations of  $BM(\mathbf{k}, H, R)$  have been carried out only in a few cases, namely: for Sweedler's Hopf algebra  $H_4$  with respect to the  $R$ -matrix  $R_0$  in [22] and for the remaining  $R$ -matrices in [6]; for the Hopf algebras of type  $H_\nu$  and all  $R$ -matrices in [7], for the group algebra of the dihedral group in [8] and for the Hopf algebras of type  $E(n)$  and all triangular  $R$ -matrices in [9]. A key role in these computations was played by  $H$ -cleft extensions of the base field  $\mathbf{k}$ .

An  $H$ -cleft extension  $B$  of  $\mathbf{k}$  is a right  $H$ -comodule algebra such that  $B^{\text{co}(H)} = \mathbf{k}$  and such that there exists a convolution invertible map  $\gamma: H \rightarrow B$  (cfr. [10]). It is well-known that cleft extensions of  $\mathbf{k}$  are parametrized by 2-cocycles, i.e., convolution invertible elements  $\sigma$  of  $(H \otimes H)^*$  satisfying the relations:

$$\sigma(h \otimes 1) = \sigma(1 \otimes h) = \varepsilon(h)$$

$$\sum \sigma(k_{(1)} \otimes m_{(1)}) \sigma(h \otimes k_{(2)} m_{(2)}) = \sum \sigma(h_{(1)} \otimes k_{(1)}) \sigma(h_{(2)} k_{(2)} \otimes m)$$

for every  $h, k, m \in H$ . The cleft extension corresponding to  $\sigma$  is isomorphic to the crossed product  ${}_\sigma H = \mathbf{k} \#_\sigma H$  that is: the comodule algebra coinciding with  $H$  as a comodule and with product given by  $h \cdot k = \sum \sigma(h_{(1)} \otimes k_{(1)}) h_{(2)} k_{(2)}$ .

Two cocycles  $\sigma$  and  $\omega$  are called cohomologous if there exists a convolution invertible element  $\theta$  in  $H^*$  for which

$$\sigma(h \otimes k) = \omega^\theta(h \otimes k) = \sum \theta(h_{(1)}) \theta(k_{(1)}) \omega(h_{(2)} \otimes k_{(2)}) \theta^{-1}(h_{(3)} k_{(3)}).$$

Two cleft extensions are equivalent if and only if they correspond to cohomologous cocycles.

The cleft extension  ${}_\sigma H$  is also a left comodule algebra for Doi's twisted Hopf algebra  ${}_\sigma H_{\sigma^{-1}}$ . The latter is obtained with the procedure dual to Drinfeld's twist and it is equal to  $H$  as a coalgebra but with product:

$$h \cdot_\sigma k = \sum \sigma(h_{(1)} \otimes k_{(1)}) h_{(2)} k_{(2)} \sigma^{-1}(h_{(3)} \otimes k_{(3)}).$$

It is well-known that if  $H$  is dual quasitriangular with universal  $r$ -form  $r$  then  $(\sigma\tau) * r * \sigma^{-1}$  is a universal  $r$ -form for  ${}_\sigma H_{\sigma^{-1}}$ . By the dual version

of [15, Proposition 2.3.5], cohomologous cocycles yield isomorphic twisted Hopf algebras and if  $H$  is dual quasitriangular, they yield isomorphic dual quasitriangular Hopf algebras.

## 2 The Main result

Given a Hopf algebra  $H$ , its opposite algebra  $H^{op}$  with its coproduct  $\Delta$  is a left and right  $H^{op}$ -comodule algebra. If  $\sigma$  is a left 2-cocycle for  $H$  then  $\sigma\tau$  is a left 2-cocycle for  $H^{op}$  and  $A_\sigma := {}_{\sigma\tau}H^{op}$  is again a right  $H^{op}$ -comodule algebra, with product:

$$h \cdot k = \sum \sigma(k_{(1)} \otimes h_{(1)})k_{(2)}h_{(2)}$$

for every  $h$  and  $k \in A_\sigma$ , so that

$$hk = \sum \sigma^{-1}(h_{(1)} \otimes k_{(1)})k_{(2)} \cdot h_{(2)}.$$

If  $H$  is dual quasitriangular with universal  $r$ -form  $r$ , the product  $\bullet$  in  $\overline{A}_\sigma$ , the opposite algebra with respect to  $r$ , is given by

$$\begin{aligned} a \bullet b &= \sum b_{(1)} \cdot a_{(1)} r(b_{(2)} \otimes a_{(2)}) \\ &= \sum ((\sigma\tau) * r)(b_{(1)} \otimes a_{(1)}) b_{(2)} a_{(2)}. \end{aligned}$$

One may wonder when  $A_\sigma$  is an  $(H, r)$ -Azumaya algebra. In this particular case, the maps  $F$  and  $G$  in Section 1 are:

$$\begin{aligned} F(h \# k)(l) &= \sum h \cdot (l_{(1)} \cdot k_{(1)}) r(l_{(2)} \otimes k_{(2)}) \\ &= \sum h \cdot \sigma(k_{(1)} \otimes l_{(1)}) k_{(2)} l_{(2)} r(l_{(3)} \otimes k_{(3)}) \\ &= \sum h \cdot \sigma(k_{(1)} \otimes l_{(1)}) r(l_{(2)} \otimes k_{(2)}) l_{(3)} k_{(3)} \\ &= \sum h \cdot (\sigma\tau * r)(l_{(1)} \otimes k_{(1)}) \sigma^{-1}(l_{(2)} \otimes k_{(2)}) k_{(3)} \cdot l_{(3)} \\ &= \sum (\sigma\tau * r * \sigma^{-1})(l_{(1)} \otimes k_{(1)}) h \cdot k_{(2)} \cdot l_{(2)} \end{aligned}$$

and

$$\begin{aligned} G(h \# k)(l) &= \sum r(h_{(2)} \otimes l_{(2)}) h_{(1)} \cdot l_{(1)} \cdot k \\ &= \sum \sigma(l_{(1)} \otimes h_{(1)}) (l_{(2)} h_{(2)}) r(h_{(3)} \otimes l_{(3)}) \cdot k \\ &= \sum \sigma(l_{(1)} \otimes h_{(1)}) r(h_{(2)} \otimes l_{(2)}) (h_{(3)} l_{(3)}) \cdot k \\ &= \sum (\sigma\tau * r * \sigma^{-1})(h_{(1)} \otimes l_{(1)}) l_{(2)} \cdot h_{(2)} \cdot k. \end{aligned}$$

The bijectivity of these maps is strictly related to the behaviour of the universal  $r$ -form  $r_\sigma = (\sigma\tau) * r * \sigma^{-1}$  in the twisted Hopf algebra  ${}_\sigma H_{\sigma^{-1}}$ . It is well-known that if  $r$  is a universal  $r$ -form for  $H$ , the map

$$\begin{aligned}\theta_r: H^{op} &\longrightarrow H^* \\ h &\mapsto r(- \otimes h)\end{aligned}$$

is a Hopf algebra homomorphism. In particular we will relate the bijectivity of the map

$$\begin{aligned}\theta_\sigma = \theta_{r_\sigma}: ({}_\sigma H_{\sigma^{-1}})^{op} &\longrightarrow ({}_\sigma H_{\sigma^{-1}})^* \\ h &\mapsto r_\sigma(- \otimes h)\end{aligned}$$

to the bijectivity of  $F$  and  $G$ . We shall follow the lines of the proof of [4, Theorem 12.4.5]. In terms of  $\theta_\sigma$  we have:

$$\begin{aligned}F(h \# k)(l) &= h \cdot \left( \sum \langle \theta_\sigma(k_{(1)}), l_{(1)} \rangle k_{(2)} \cdot l_{(2)} \right) \\ G(h \# k)(l) &= \left( \sum \langle \theta_\sigma(l_{(1)}), h_{(1)} \rangle l_{(2)} \cdot h_{(2)} \right) \cdot k.\end{aligned}$$

We recall that for a finite-dimensional Hopf algebra  $H$  the space of left integrals  $\int_{H^*}^l$  for  $H^*$  is one-dimensional  $\mathbf{k}\zeta$ , say. As a consequence of the Fundamental Theorem for Hopf modules there is a  $\mathbf{k}$ -linear isomorphism

$$\begin{aligned}V: \int_{H^*}^l \otimes H &\longrightarrow H^* \\ V(\xi \otimes h)(k) &= \xi(kS(h)).\end{aligned}$$

It is well-known that if we put  $v(h) := V(\zeta \otimes h)$  the following formula holds:

$$\sum \langle v(h), k_{(2)} \rangle k_{(1)} = \sum \langle v(h_{(1)}), k \rangle h_{(2)} \quad (2.1)$$

for every  $h, k \in H$ .

Let us denote by  $w(h) := (S^{-1})^*(v(h)) \in H^*$  for every  $h \in H$ . Then one has:

$$\sum \langle w(h), k_{(1)} \rangle S^{-1}(k_{(2)}) = \sum \langle w(h_{(1)}), k \rangle h_{(2)}. \quad (2.2)$$

Applying the antipode  $S$  on both sides we get:

$$\sum \langle w(h), k_{(1)} \rangle k_{(2)} = \sum \langle w(h_{(1)}), k \rangle S(h_{(2)}) \quad (2.3)$$

which is the counterpart of (2.1) for  $H^{op, cop}$ .

We introduce the following maps  $S_i: A_\sigma \longrightarrow A_\sigma$  for  $i = 1, 2$ :

$$S_1(h) = \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)})S(h_{(1)});$$

$$S_2(h) = \sum \sigma^{-1}(h_{(3)} \otimes S^{-1}(h_{(2)}))S^{-1}(h_{(1)}).$$

A straightforward computation yields:

$$\sum h_{(2)} \cdot S_1(h_{(1)}) = \varepsilon(h) = \sum S_2(h_{(1)}) \cdot h_{(2)} \quad (2.4)$$

for every  $h \in A_\sigma$ . Besides, by the left cocycle condition we have:

$$\sigma(k \otimes lm) = \sum \sigma^{-1}(l_{(1)} \otimes m_{(1)})\sigma(k_{(1)} \otimes l_{(2)})\sigma(k_{(2)}l_{(3)} \otimes m_{(2)}) \quad (2.5)$$

and

$$\sigma(kl \otimes m) = \sum \sigma^{-1}(k_{(1)} \otimes l_{(1)})\sigma(l_{(2)} \otimes m_{(1)})\sigma(k_{(2)} \otimes l_{(3)}m_{(2)}). \quad (2.6)$$

Applying (2.5) to  $k = h_{(1)}$ ,  $l = S(h_{(2)})$  and  $m = h_{(3)}$  and adding all terms we get:

$$\varepsilon(h) = \sum \sigma^{-1}(S(h_{(3)}) \otimes h_{(4)})\sigma(h_{(1)} \otimes S(h_{(2)})).$$

This formula was already observed, in greater generality, by Blattner and Montgomery, see [16, Proposition 7.2.7]. It implies that

$$\sum S_1(h_{(2)}) \cdot h_{(1)} = \varepsilon(h) \quad (2.7)$$

for every  $h \in A_\sigma$ . Applying (2.6) to  $k = h_{(3)}$ ,  $l = S^{-1}(h_{(2)})$  and  $m = h_{(1)}$  and adding all terms we have:

$$\varepsilon(h) = \sum \sigma^{-1}(h_{(4)} \otimes S^{-1}(h_{(3)}))\sigma(S^{-1}(h_{(2)}) \otimes h_{(1)})$$

and this implies that

$$\sum h_{(1)} \cdot S_2(h_{(2)}) = \varepsilon(h). \quad (2.8)$$

Now we are ready to state the main result of this section.

**Theorem 2.1** *Let  $H$  be finite-dimensional dual quasitriangular Hopf algebra  $H$  with universal  $r$ -form  $r$ . Let  $\sigma$  be a left 2-cocycle for  $H$ . Then, the algebra  $A_\sigma$  is  $(H, r)$ -Azumaya if and only if  $\theta_\sigma$  is invertible.*



**Proof:** Let  $\theta_\sigma$  be invertible. We shall see that, for every  $\eta \in A_\sigma^*$  and every  $m \in A_\sigma$  the endomorphism of  $A_\sigma$  given by  $h \mapsto \langle \eta, h \rangle m$  belongs to the image of  $F$ . Let  $h \in H$  be such that  $\eta = w(h)$  and let us consider the following element of  $A_\sigma \# \overline{A_\sigma}$ :

$$\Gamma = \sum m \cdot S_2(S(h_{(2)})) \cdot S_1(\theta_\sigma^{-1}((w(h_{(1)}))_{(2)})) \# \theta_\sigma^{-1}((w(h_{(1)}))_{(1)}).$$

Then for every  $l \in A_\sigma$

$$\begin{aligned} F(\Gamma)(l) &= \sum \langle \theta_\sigma(\theta_\sigma^{-1}((w(h_{(1)}))_{(1)})), l_{(1)} \rangle m \cdot S_2(S(h_{(2)})) \cdot \\ &\quad S_1(\theta_\sigma^{-1}((w(h_{(1)}))_{(3)})) \cdot \theta_\sigma^{-1}((w(h_{(1)}))_{(2)}) \cdot l_{(2)} \\ &= \sum \langle (w(h_{(1)}))_{(1)}, l_{(1)} \rangle m \cdot S_2(S(h_{(2)})) \cdot \varepsilon((w(h_{(1)}))_{(2)}) l_{(2)} \\ &= \sum \langle w(h_{(1)}), l_{(1)} \rangle m \cdot S_1(S(h_{(2)})) \cdot l_{(2)} \end{aligned}$$

Applying (2.3) we have

$$\begin{aligned} F(\Gamma)(l) &= \sum \langle w(h_{(1)}), l \rangle m \cdot S_2(S(h_{(3)})) \cdot S(h_{(2)}) \\ &= \sum \langle w(h_{(1)}), l \rangle m \cdot S_2((S(h_{(2)}))_{(1)}) \cdot (S(h_{(2)}))_{(2)} \\ &= \sum \langle w(h_{(1)}), l \rangle \varepsilon(S(h_{(2)})) m = \langle w(h), l \rangle m \end{aligned}$$

where we used (2.4). Hence  $F$  is surjective. Similarly, let  $(\theta_\sigma^{-1})^*$  be the dual map of  $\theta_\sigma^{-1}$  with respect to the non-degenerate pairing  $\langle \cdot, \cdot \rangle$ . The map  $(\theta_\sigma^{-1})^*$  is a well-defined Hopf algebra map  $({}_\sigma H_{\sigma^{-1}})^{*, \text{cop}} \longrightarrow {}_\sigma H_{\sigma^{-1}}$  and it is bijective if  $\theta_\sigma$  is so. Let  $h$  and  $m$  be as before and let  $\Gamma'$  be the following element of  $A_\sigma \# \overline{A_\sigma}$ :

$$\Gamma' = \sum (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)}) \# S_2((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)})) \cdot S_1(S(h_{(2)})) \cdot m.$$

Then we have:

$$\begin{aligned} G(\Gamma')(l) &= \sum \langle \theta_\sigma(l_{(1)}), ((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)}))_{(1)} \rangle l_{(2)} \cdot \\ &\quad ((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)}))_{(2)} \cdot S_2((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)})) \cdot S_1(S(h_{(2)})) \cdot m \\ &= \sum \langle \theta_\sigma(l_{(1)}), (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)}))_{(1)} \rangle l_{(2)} \cdot \\ &\quad (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)}))_{(2)} \cdot S_2((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)})) \cdot S_1(S(h_{(2)})) \cdot m \\ &= \sum \langle \theta_\sigma(l_{(1)}), (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(3)})) \rangle l_{(2)} \cdot \\ &\quad (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)}) \cdot S_2((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)})) \cdot S_1(S(h_{(2)})) \cdot m \\ &= \sum \langle \theta_\sigma(l_{(1)}), (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)})) \rangle l_{(2)} \cdot ((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)}))_{(2)} \cdot \\ &\quad \cdot S_2((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)}))_{(1)} \cdot S_1(S(h_{(2)})) \cdot m \\ &= \sum \langle \theta_\sigma(l_{(1)}), (\theta_\sigma^{-1})^*((w(h_{(1)}))_{(2)})) \rangle l_{(2)} \cdot ((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)}))_{(1)} \cdot \\ &\quad S_2((\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)}))_{(2)} \cdot S_1(S(h_{(2)})) \cdot m \\ &= \sum \langle w(h_{(1)}), l_{(1)} \rangle l_{(2)} \cdot S_1(S(h_{(2)})) \cdot m \end{aligned}$$

where for the last equality we used (2.8) applied to  $(\theta_\sigma^{-1})^*((w(h_{(1)}))_{(1)})$ . By (2.3) we get:

$$\begin{aligned} G(\Gamma')(l) &= \sum \langle w(h_{(1)}), l \rangle S(h_{(2)}) \cdot S_1(S(h_{(3)})) \cdot m \\ &= \sum \langle w(h_{(1)}), l \rangle (S(h_{(2)}))_{(2)} \cdot S_1((S(h_{(2)}))_{(1)}) \cdot m \\ &= \sum \langle w(h), l \rangle m \end{aligned}$$

where the last equality follows from (2.4). Therefore, if  $\theta_\sigma$  is bijective then  $F$  and  $G$  are surjective, hence bijective.

Let us now assume that  $F$  is bijective. We will show that  $\theta_\sigma$  is surjective. We recall that if  $A$  is a right  $H$ -comodule,  $\text{End}(A)$  with comodule structure given by (1.1) is isomorphic, as right comodule, to  $A \otimes A^*$  with comodule structure on  $A^*$  given by

$$\rho(\xi)(a) = \sum \langle \xi, a_{(1)} \rangle S^{-1}(a_{(2)})$$

for every  $\xi \in A^*$  and every  $a \in A$ . An isomorphism  $\Phi$  is given by  $\Phi(a \otimes \xi)(b) = \langle \xi, b \rangle a$  for every  $a, b \in A$  and every  $\xi \in A^*$ .

Let us consider  $\eta \in H^*$ . We shall show that  $\eta$  belongs to the image of  $\theta_\sigma$ . We know that  $\eta = w(h)$  for some  $h$ . For  $A = A_\sigma \cong H$  as comodule,  $\Phi(1 \otimes \eta)$  belongs to the image of  $F$ , hence, there exists a  $\Gamma = \sum a_i \# b_i \in A_\sigma \# \overline{A_\sigma}$  for which  $F(\Gamma) = 1 \otimes \eta = 1 \otimes w(h)$ . Since  $F$  is a comodule map, there holds:

$$F(\sum a_{i1} \# b_{i1}) \otimes b_{i2} a_{i2} = \sum 1 \otimes \eta_{(1)} \otimes \eta_{(2)}$$

that is, for every  $k \in H$

$$\begin{aligned} \sum \langle \theta_\sigma(b_{i(1)}), k_{(1)} \rangle a_{i(1)} \cdot b_{i(2)} \cdot k_{(2)} \otimes b_{i(3)} a_{i(2)} &= \sum \langle \eta, k_{(1)} \rangle S^{-1}(k_{(2)}) = \\ \sum \langle w(h), k_{(1)} \rangle S^{-1}(k_{(2)}) &= \sum \langle w(h_{(1)}), k \rangle h_{(2)} \end{aligned}$$

where the last equality follows from (2.2). Applying the linear operator  $(S \otimes \text{id})\tau$  to the last and the first term of the above chain of equalities we obtain:

$$\begin{aligned} \sum \langle w(h_{(1)}), k \rangle S(h_{(2)}) \otimes 1 &= \sum S(a_{i(2)}) S(b_{i(3)}) \otimes \langle \theta_\sigma(b_{i(1)}), k_{(1)} \rangle a_{i(1)} \cdot b_{i(2)} \cdot k_{(2)} \\ &= \sum \sigma^{-1}((S(a_{i(2)}))_{(1)} \otimes (S(b_{i(3)}))_{(1)}) (S(b_{i(3)}))_{(2)} \cdot (S(a_{i(2)}))_{(2)} \otimes \\ &\quad \langle \theta_\sigma(b_{i(1)}), k_{(1)} \rangle a_{i(1)} \cdot b_{i(2)} \cdot k_{(2)} \\ &= \sum \sigma^{-1}(S(a_{i(3)}) \otimes S(b_{i(4)})) S(b_{i(3)}) \cdot S(a_{i(2)}) \otimes \\ &\quad \langle \theta_\sigma(b_{i(1)}), k_{(1)} \rangle a_{i(1)} \cdot b_{i(2)} \cdot k_{(2)}. \end{aligned}$$

Applying the product in  $A_\sigma$  on the first and the last term of the above chain of equalities we obtain:

$$\sum \langle w(h_{(1)}), k \rangle S(h_{(2)}) = \sum \sigma^{-1}(S(a_{i(3)}) \otimes S(b_{i(4)})) S(b_{i(3)}) \cdot S(a_{i(2)}) \cdot \langle \theta_\sigma(b_{i(1)}), k_{(1)} \rangle a_{i(1)} \cdot b_{i(2)} \cdot k_{(2)}.$$

A direct computation yields, for every  $l \in H$ :

$$\sum S(l_{(2)}) \cdot l_{(1)} = \sum \sigma(l_{(1)} \otimes S(l_{(4)})) l_{(2)} S(l_{(3)}) = \sum \sigma(l_{(1)} \otimes S(l_{(2)})).$$

Using this formula in the previous equality we have:

$$\sum \langle w(h_{(1)}), k \rangle S(h_{(2)}) = \sum \sigma^{-1}(S(a_{i(3)}) \otimes S(b_{i(4)})) \sigma(b_{i(2)} \otimes S(b_{i(3)})) \sigma(a_{i(1)} \otimes S(a_{i(2)})) \langle \theta_\sigma(b_{i(1)}), k_{(1)} \rangle k_{(2)}.$$

Applying  $\varepsilon$  on both sides and observing that the equality holds for every  $k$  yields:

$$\eta = \theta_\sigma \left( \sum \sigma^{-1}(S(a_{i(3)}) \otimes S(b_{i(4)})) \sigma(b_{i(2)} \otimes S(b_{i(3)})) \sigma(a_{i(1)} \otimes S(a_{i(2)})) (b_{i(1)}) \right).$$

Hence,  $\eta \in \text{Im}(\theta_\sigma)$  for every  $\eta \in H^*$ .

Let us now suppose that  $G$  is surjective. In a similar fashion we shall prove that  $\theta_\sigma^*$  is surjective. The right  $H$ -comodule  $\text{End}(A)^{op}$  with comodule structure given by (1.2) is isomorphic to the right  $H$ -comodule  $A^* \otimes A$  with comodule structure on  $A^*$  given by

$$\rho(\xi)(a) = \sum \langle \xi, a_{(1)} \rangle S(a_{(2)})$$

for every  $\xi \in A^*$  and every  $a \in A$ . An isomorphism is given by  $\Psi(\xi \otimes a)(b) = \langle \xi, b \rangle a$  for every  $a, b \in A$  and every  $\xi \in A^*$ .

Let  $u(m) = S^*(v(m))$  for every  $m \in H$ , let  $\eta \in H^*$  and let  $h$  be such that  $\eta = u(h)$ . As before  $\Psi(\eta \otimes 1) = G(\Gamma')$  for some  $\Gamma' = \sum c_i \# d_i \in A_\sigma \# \overline{A_\sigma}$ . Since  $G$  is a comodule map, there holds:

$$G(\sum c_{i(1)} \# d_{i(1)}) \otimes d_{i(2)} c_{i(2)} = \sum \eta_{(1)} \otimes 1 \otimes \eta_{(2)}$$

that is, for every  $k \in H$ :

$$\begin{aligned} \sum \langle \theta_\sigma(k_{(1)}), c_{i(1)} \rangle k_{(2)} \cdot c_{i(2)} \cdot d_{i(1)} \otimes d_{i(2)} c_{i(3)} &= \sum \langle \eta, k_{(1)} \rangle 1 \otimes S(k_{(2)}) \\ &= \sum \langle u(h), k_{(1)} \rangle 1 \otimes S(k_{(2)}) = \sum \langle v(h), (S(k))_{(2)} \rangle 1 \otimes (S(k))_{(1)} \\ &= \sum \langle u(h_{(1)}), k \rangle 1 \otimes h_{(2)}. \end{aligned}$$

Applying the linear operator  $(\text{id} \otimes S^{-1})$  to the first and the last term of the above chain of equalities we obtain:

$$\begin{aligned} \sum \langle u(h_{(1)}), k \rangle 1 \otimes S^{-1}(h_{(2)}) &= \sum \langle \theta_\sigma(k_{(1)}), c_{i(1)} \rangle k_{(2)} \cdot c_{i(2)} \cdot d_{i(1)} \otimes S^{-1}(c_{i(3)}) S^{-1}(d_{i(2)}) \\ &= \sum \langle \theta_\sigma(k_{(1)}), c_{i(1)} \rangle k_{(2)} \cdot c_{i(2)} \cdot d_{i(1)} \otimes \\ &\quad S^{-1}(d_{i(2)}) \cdot S^{-1}(c_{i(3)}) \sigma^{-1}(S^{-1}(c_{i(4)}) \otimes S^{-1}(d_{i(3)})). \end{aligned}$$

Applying the product in  $A_\sigma$  on the first and the last term of the above chain of equalities and using the formula:

$$\sum l_{(1)} \cdot S^{-1}(l_{(2)}) = \sum \sigma(S^{-1}(l_{(4)}) \otimes l_{(1)}) S^{-1}(l_{(3)}) l_{(2)} = \sum \sigma(S^{-1}(l_{(2)}) \otimes l_{(1)})$$

for every  $l \in H$  we obtain:

$$\begin{aligned} \sum \langle u(h_{(1)}), k \rangle S^{-1}(h_{(2)}) &= \sum \langle \theta_\sigma(k_{(1)}), c_{i(1)} \rangle \sigma(S^{-1}(d_{i(2)}) \otimes d_{i(1)}) \\ &\quad \sigma(S^{-1}(c_{i(3)}) \otimes c_{i(2)}) \sigma^{-1}(S^{-1}(c_{i(4)}) \otimes S^{-1}(d_{i(3)})) k_{(2)}. \end{aligned}$$

Applying  $\varepsilon$  on both sides and observing that equality holds for every  $k$  yields  $\eta = \theta_\sigma^*(z)$  with

$$z = \sum \sigma(S^{-1}(d_{i(2)}) \otimes d_{i(1)}) \sigma(S^{-1}(c_{i(3)}) \otimes c_{i(2)}) \sigma^{-1}(S^{-1}(c_{i(4)}) \otimes S^{-1}(d_{i(3)})) c_{i(1)}$$

whence the proof.  $\square$

**Corollary 2.2** *Let  $H$  be a finite-dimensional dual quasitriangular Hopf algebra with universal  $r$ -form  $r$ . Then the Hopf algebra  $H^{op}$  is  $(H, r)$ -Azumaya if and only if  $\theta_r$  is bijective.*  $\square$

**Remark 2.3** Let us observe that, as a consequence of the proof of Theorem 2.1,  $F$  is bijective if and only if  $G$  is so.

**Remark 2.4** The condition in Theorem 2.1 that the Hopf algebra is over a field could be relaxed. Indeed the proof would work by localization, just as in [4, Theorem 12.4.5], for any faithfully projective Hopf algebra with bijective antipode over a commutative ring.

**Remark 2.5** Let  $(H, r)$  be dual quasitriangular. Then  $(H^{op}, r\tau)$  is again dual quasitriangular. Theorem 2.1 for  $H^{op}$  states that if  $\sigma$  is a 2-cocycle for  $H^{op}$  then  ${}_{\sigma\tau}H$  is  $(H^{op}, r\tau)$ -Azumaya if and only if the map:

$$\theta_\sigma: H \longrightarrow (H^{op})^*$$

$$h \mapsto r_\sigma(h \otimes -)$$

is an isomorphism. If  $H$  is commutative and cocommutative, we recover [4, Theorem 12.4.5] with  $f = \sigma\tau$  and  $\theta: H \longrightarrow H^*$  given by  $\theta(h) = r\tau(h \otimes -)$ . Indeed, the action of  $H$  determined by the pairing  $r\tau$  is

$$h \rightharpoonup k = \sum k_{(1)} r(k_{(2)} \otimes h) = \sum k_{(1)} \langle \theta(k_{(2)}), h \rangle.$$

Let us observe that our  $\theta_\sigma$  is Caenepeel's  $(\theta * d)^*$ .

## 2.1 The dual picture

We would like to outline briefly the dual picture, i.e., the analysis of  $(H, R)$ -Azumaya algebras for a quasitriangular Hopf algebra  $H$  with  $R$ -matrix  $R$ . In order to fix notation we recall well-known facts about isomorphisms of Brauer groups and the standard equivalence  $\mathcal{D}$  between the category  ${}_H\mathcal{M}$  and the category  $\mathcal{M}^{H^*}$  (see for instance [16, Lemma 1.6.4]). Let  $\otimes^{op}$  denote the functor obtained from  $\otimes$  by reversing the order of the tensorands. The functor  $\mathcal{D}$ , together with the natural transformations  $\tau_{UV}: \mathcal{D}(U) \otimes^{op} \mathcal{D}(V) \rightarrow \mathcal{D}(U \otimes V)$  for every pair of objects  $U, V$  in  ${}_H\mathcal{M}$ , and with  $\text{id}_U: \mathbf{k} \rightarrow \mathcal{D}(\mathbf{k}) = \mathbf{k}$ , define an equivalence of monoidal categories between  $({}_H\mathcal{M}, \otimes, \mathbf{k})$  and  $(\mathcal{M}^{H^*}, \otimes^{op}, \mathbf{k})$ . If  $H$  is quasitriangular with  $R$ -matrix  $R = \sum R^{(1)} \otimes R^{(2)}$  then  $H^*$  is dual quasitriangular with universal  $r$ -form  $R$ , viewed as an element of  $(H \otimes H)^{**}$ . The functor  $\mathcal{D}$  together with  $\tau$  and  $\text{id}$  define an equivalence of braided monoidal categories between  $(\mathcal{M}^{H^*}, \otimes, \mathbf{k})$  and  $({}_H\mathcal{M}, \otimes^{op}, \mathbf{k})$ . Here the braiding in  $(\mathcal{M}^{H^*}, \otimes, \mathbf{k})$  is given by  $(u \otimes v) \mapsto \sum R^{(2)}.v \otimes R^{(1)}.u$ , the braiding in  $({}_H\mathcal{M}, \otimes^{op}, \mathbf{k})$  is given by  $m \otimes n \mapsto \sum n_{(0)} \otimes m_{(0)} \langle n_{(1)}, R^{(1)} \rangle \langle m_{(1)}, R^{(2)} \rangle$  and the braiding in  $(\mathcal{M}^{H^*}, \otimes^{op}, \mathbf{k})$  is induced by the braiding in  $(\mathcal{M}^{H^*}, \otimes, \mathbf{k})$ . The reversed equivalence induces an isomorphism

$$BM(\mathbf{k}, H, R) \mapsto BC(\mathbf{k}, H^*, R)^{op} \cong BC(\mathbf{k}, H^*, R)$$

where the class of  $A$  with given  $H$ -module structure is mapped to the class of  $A^{op}$  with right  $H^{*,op}$ -comodule structure determined by the functor  $\mathcal{D}$ .

The dual version of Theorem 2.1 reads:

**Corollary 2.6** *Let  $H$  be a finite-dimensional quasitriangular Hopf algebra with  $R$ -matrix  $R$ . Let  $C = \sum C^{(1)} \otimes C^{(2)} \in H \otimes H$  be a cocycle for  $H^*$  and let  $R_C = (\tau C)RC^{-1}$ . Then  $\mathbf{k} \#_C H^* = {}_C H^*$  with  $H$ -action:  $h \rightharpoonup f = \sum f_{(1)} \langle f_{(2)}, h \rangle$  is  $(H, R)$ -Azumaya if and only if the map:  $\theta: H^{*,op} \longrightarrow H$  given by  $\theta(f) = \sum R_C^{(1)} \langle f, R_C^{(2)} \rangle$  is an isomorphism.  $\square$*

In particular

**Corollary 2.7** *Let  $H$  be a finite-dimensional quasitriangular Hopf algebra with  $R$ -matrix  $R$ . Then  $H^*$  with  $H$ -action:  $h \rightharpoonup f = \sum f_{(1)} \langle f_{(2)}, h \rangle$  is  $(H, R)$ -Azumaya if and only if the map:  $\theta: H^{*,op} \longrightarrow H$  given by  $\theta(f) = \sum R^{(1)} \langle f, R^{(2)} \rangle$  is an isomorphism.  $\square$*

### 3 An Example: $E(n)$

Let  $\text{char}(\mathbf{k}) \neq 2$ , let  $n \geq 1$  and let  $E(n)$  denote the Hopf algebra generated by  $c$  and  $x_i$  for  $1 \leq i \leq n$  with relations:

$$c^2 = 1, \quad cx_i + x_i c = 0; \quad x_i x_j + x_j x_i = 0, \quad x_i^2 = 0$$

coproduct:

$$\Delta(c) = c \otimes c; \quad \Delta(x_i) = 1 \otimes x_i + x_i \otimes c$$

and antipode  $S(c) = c$  and  $S(x_j) = cx_j$ . The Hopf algebra  $E(n)$  is quasitriangular, isomorphic to its opposite and self-dual. Its  $R$ -matrices were classified in [18] and they are parametrized by matrices in  $M_n(\mathbf{k})$ . By self-duality, the universal  $r$ -forms are parametrized by elements of  $M_n(\mathbf{k})$  and they are given as follows: for a matrix  $A = (a_{ij}) \in M_n(\mathbf{k})$  and for  $s$ -tuples  $P, F$  of increasing elements in  $\{1, \dots, n\}$  we define  $|P| = |F| = s$  and  $x_P$  as the product of the  $x_j$ 's whose index belongs to  $P$ , taken in increasing order. Any bijective map  $\eta: F \rightarrow F$  may be identified with an element of the symmetric group  $S_s$ . Let  $\text{sign}(\eta)$  denote the signature of  $\eta$ . If  $P = \emptyset$  then we take  $F = \emptyset$  and  $\text{sign}(\eta) = 1$ . Finally, by  $a_{P, \eta(F)}$  we denote the product  $a_{p_1, f_{\eta(1)}} \cdots a_{p_s, f_{\eta(s)}}$ . For  $P = \emptyset$  we define  $a_{P, \eta(F)} := 1$ . Then the universal  $r$ -form corresponding to the matrix  $A$  is:

$$r_A = \sum_P (-1)^{\frac{|P|(|P|-1)}{2}} \sum_{F, |F|=|P|, \eta \in S_{|P|}} \text{sign}(\eta) a_{P, \eta(F)} ((x_P)^* \otimes (x_F)^* + (cx_P)^* \otimes (x_F)^* + (-1)^{|P|} (x_P)^* \otimes (cx_F)^* - (-1)^{|P|} (cx_P)^* \otimes (cx_F)^*).$$

In particular,  $r_A(x_i \otimes x_j) = a_{ij}$ .

The  $E(n)$ -cleft extensions of  $\mathbf{k}$  up to equivalence were classified in [19]. They are parametrized by an invertible scalar  $\alpha$ , a vector  $\gamma \in \mathbf{k}^n$  and a lower triangular  $n \times n$  matrix  $\Lambda = (\lambda_{ij})$ . On the generators the corresponding cocycle  $\sigma = \sigma(\alpha, \gamma, \Lambda)$  has values:

$$\sigma(c \otimes c) = \alpha; \quad \sigma(x_i \otimes c) = \gamma_i; \quad \sigma(c \otimes x_i) = 0; \quad \sigma(x_i \otimes x_j) = \lambda_{ij}.$$

The cleft extension corresponding to the cocycle  $\sigma$  is the generalized Clifford algebra  $Cl(\alpha, \gamma, \lambda)$  with generators  $u$  and  $v_i$ , for  $i = 1, \dots, n$ , relations

$$u^2 = 1, \quad uv_i + v_iu = \gamma_i, \quad v_j^2 = \lambda_{jj}, \quad v_iv_j + v_jv_i = \lambda_{ij} \quad \text{for } i \neq j \quad (3.1)$$

and with comodule algebra structure given by:

$$\rho(u) = u \otimes c, \quad \rho(v_j) = 1 \otimes x_j + v_j \otimes c. \quad (3.2)$$

It is clear that  ${}_{\sigma\tau}E(n)^{op} = Cl(\alpha, \gamma, \Lambda)^{op} \cong Cl(\alpha, \gamma, \Lambda)$ . We shall apply Theorem 2.1 to  $Cl(\alpha, \gamma, \lambda)$  and reduce the question on when this comodule algebra is  $(E(n), r_A)$ -Azumaya to a simple linear algebra problem. Since  $E(n) = ({}_{\sigma}E(n)_{\sigma^{-1}})^{op}$  as coalgebras, by [12, Proposition 2.4.2] the coalgebra map  $\theta_{\sigma}: ({}_{\sigma}E(n)_{\sigma^{-1}})^{op} \rightarrow ({}_{\sigma}E(n)_{\sigma^{-1}})^*$  is injective if and only if its restriction to the span  $W$  of  $1, c$  the  $x'_j$ 's and the  $cx_j$ 's is injective. For every  $h \in E(n)$  let us denote the corresponding element in the twisted Hopf algebra  ${}_{\sigma}E(n)_{\sigma^{-1}}$  by  $\bar{h}$ . If for every  $\bar{h} \in W$  the restriction of the functional  $\theta_{\sigma}(\bar{h})$  to  $W$  is not identically zero, then  $\theta_{\sigma}$  is injective. Let us now assume that there exists an element  $\bar{h} \in W$  such that  $\theta_{\sigma}(\bar{h})(\bar{k}) = 0$  for every  $\bar{k} \in W$ . Let

$$\bar{h} = e + f\bar{c} + \sum s_i\bar{x}_i + \sum t_i\bar{c}x_i.$$

By the description of the cocycles in [19] we have:

$$\sigma(cx_i \otimes c) = \gamma_i; \quad \sigma(cx_i \otimes x_j) = \lambda_{ij};$$

$$\sigma(c \otimes cx_i) = 0; \quad \sigma(x_i \otimes cx_j) = -\lambda_{ij}; \quad \sigma(cx_i \otimes cx_j) = -\alpha\lambda_{ij}$$

and

$$\begin{aligned} \sigma^{-1}(c \otimes c) &= \alpha^{-1}; & \sigma^{-1}(c \otimes x_j) &= 0; & \sigma^{-1}(x_j \otimes c) &= -\alpha^{-1}\gamma_j; \\ \sigma^{-1}(c \otimes cx_j) &= 0; & \sigma^{-1}(cx_j \otimes c) &= -\alpha^{-1}\gamma_j; & \sigma^{-1}(x_i \otimes x_j) &= -\alpha^{-1}\lambda_{ij}; \\ \sigma^{-1}(cx_i \otimes x_j) &= -\lambda_{ij}; & \sigma^{-1}(x_i \otimes cx_j) &= \lambda_{ij}; & \sigma^{-1}(cx_i \otimes cx_j) &= \lambda_{ij}. \end{aligned}$$

A direct computation shows that, for the universal  $r$ -form  $r_A$  and the above cocycle  $\sigma$  one has:

$$\begin{aligned} r_{A,\sigma}(x_j \otimes c) &= -\alpha^{-1}\gamma_j, & r_{A,\sigma}(c \otimes x_j) &= -\alpha^{-1}\gamma_j, & r_{A,\sigma}(c \otimes cx_j) &= \gamma_j, \\ r_{A,\sigma}(cx_i \otimes cx_j) &= \alpha b_{ij}, & r_{A,\sigma}(x_i \otimes x_j) &= \alpha^{-1}b_{ij}, & r_{A,\sigma}(cx_j \otimes c) &= \gamma_j, \\ r_{A,\sigma}(cx_i \otimes x_j) &= b_{ij} + \alpha^{-1}\gamma_i\gamma_j, & r_{A,\sigma}(x_i \otimes cx_j) &= -b_{ij}, & r_{A,\sigma}(c \otimes c) &= -1 \end{aligned}$$

where  $B$  is the  $n \times n$  matrix whose  $(i, j)$ -entry is  $b_{ij} = a_{ij} - \lambda_{ij} - \lambda_{ji}$ .

We have:

$$\begin{aligned} \theta_\sigma(\bar{h})(1) &= e + f = 0 \\ \theta_{A,\sigma}(\bar{h})(\bar{c}) &= e + fr_{A,\sigma}(c \otimes c) + \sum_j s_j r_{A,\sigma}(c \otimes x_j) + \sum_j t_j r_{A,\sigma}(c \otimes cx_j) \\ &= e - f + \sum(t_j - \alpha^{-1} \sum s_j) \gamma_j = 0 \\ \theta_{A,\sigma}(\bar{h})(\bar{x}_i) &= -\alpha^{-1} f \gamma_i + \sum b_{ij} (\alpha^{-1} s_j - t_j) = 0 \\ \theta_{A,\sigma}(\bar{h})(\bar{c}x_i) &= f \gamma_i + \sum b_{ij} (s_j + \alpha t_j) + \alpha^{-1} \gamma_i \sum \gamma_j s_j = 0. \end{aligned}$$

This is possible if and only if there exists a non-trivial solution  $(y, \underline{x}, \underline{z})$  with  $y \in \mathbf{k}$  and  $\underline{x}, \underline{z} \in \mathbf{k}^n$  to the system:

$$\begin{cases} (\underline{z} - \alpha^{-1} \underline{x}) \bullet \gamma &= 2y; \\ B(\alpha \underline{z} - \underline{x}) &= -y\gamma; \\ B(\alpha \underline{z} + \underline{x}) &= -(y + \alpha^{-1} \underline{x} \bullet \gamma)\gamma \end{cases}$$

where  $\bullet$  denotes the usual dot product in  $\mathbf{k}^n$ . This system is equivalent to:

$$\begin{cases} \underline{z} \bullet \gamma &= 2y + \alpha^{-1} \underline{x} \bullet \gamma; \\ 2\alpha B \underline{z} &= -(2y + \alpha^{-1} \underline{x} \bullet \gamma)\gamma; \\ 2B \underline{x} &= -(\alpha^{-1} \underline{x} \bullet \gamma)\gamma \end{cases}$$

which is equivalent to:

$$\begin{cases} \underline{z} \bullet \gamma &= 2y + \alpha^{-1} \underline{x} \bullet \gamma; \\ 2\alpha B \underline{z} &= -(\underline{z} \bullet \gamma)\gamma; \\ 2\alpha B \underline{x} &= -(\underline{x} \bullet \gamma)\gamma. \end{cases}$$

If the third equation admits a non-trivial solution  $\underline{z}$ , we may take  $\underline{z} = 0$  and  $2f = 2y = -\alpha^{-1} \underline{x} \bullet \gamma$  and the system admits a non-trivial solution. If the third equation does not admit a non-trivial solution then the same holds for the second equation forcing  $\underline{x} = \underline{0} = \underline{z}$  and  $y = 0$ . Hence, if such an  $\bar{h}$  exists, we may assume that it is  $(1, c)$ -skew-primitive.

Let us observe that, due to the particular coalgebra structure of  $E(n)$ , the elements  $\bar{c}$  and  $\bar{x}_j$  for  $j = 1, \dots, n$  are algebra generators in  ${}_\sigma E(n)_{\sigma^{-1}}$ . Indeed, the elements  $\bar{c}^a \bar{x}_P$  with  $a = 0, 1$  and  $P \subset \{1, \dots, n\}$  span  ${}_\sigma E(n)_{\sigma^{-1}}$ . One can prove by induction on  $|P| = m$  that  $\bar{c} \bar{x}_P$  lies in the span of  $\bar{c} \cdot_\sigma \bar{x}_P$  and of terms of the form  $\bar{c}^b \cdot_\sigma \bar{x}_{j_1} \cdot_\sigma \dots \cdot_\sigma \bar{x}_{j_k}$  with  $J$  strictly contained in  $P$  and  $b = 0, 1$ . Similarly, it can be proved by induction on  $|P| = m$  that  $\bar{x}_P$  lies in the span of  $\bar{x}_{p_1} \cdot_\sigma \dots \cdot_\sigma \bar{x}_{p_m}$  and  $\bar{c}^b \cdot_\sigma \bar{x}_{j_1} \cdot_\sigma \dots \cdot_\sigma \bar{x}_{j_k}$  with  $J$  strictly



contained in  $P$  and  $b = 0, 1$ . Then every element  $\bar{l}$  of  ${}_{\sigma}E(n)_{\sigma^{-1}}$  is spanned by a product  $\bar{l} = \bar{k}_1 \cdot_{\sigma} \cdots \cdot_{\sigma} \bar{k}_m$  with  $k_j \in W$ . Thus

$$\begin{aligned} (\theta_{\sigma}(\bar{h}))(\bar{l}) &= \theta_{\sigma}(\bar{h})(\bar{k}_1 \cdot_{\sigma} \cdots \cdot_{\sigma} \bar{k}_m) \\ &= \sum \prod \theta_{\sigma}(\bar{h}_{(i)})(\bar{k}_i) = 0 \end{aligned}$$

because in each summand we have  $h_{(i)} = h$  for some  $i$ , since we have assumed  $h$  to be skew-primitive. Hence if  $\theta_{\sigma}(\bar{h})$  is identically zero on  $W$ , it is zero and  $\theta_{\sigma}$  is not injective. If we denote by  $\Gamma$  is the  $n \times n$  matrix whose  $(i, j)$ -entry is  $\Gamma_{ij} = \gamma_i \gamma_j$ , the previous discussion shows that  $\theta_{\sigma}$  is injective if and only if  $2\alpha B \underline{x} = -(\underline{x} \bullet \gamma)\gamma$ , and therefore

$$(2\alpha B + \Gamma)\underline{z} = \underline{0}$$

admits no non-trivial solutions. We have reached the following result:

**Proposition 3.1** *The cleft extension  $Cl(\alpha, \gamma, \Lambda)^{op}$  is  $(E(n), r_A)$ -Azumaya if and only if  $\det(2\alpha(A - \Lambda - \Lambda^t) + \Gamma) \neq 0$ .  $\square$*

It was proved in [9, Lemma 2.3] that when  $\gamma = 0$  and  $\alpha = 1$  the corresponding cocycle  $\sigma$  is lazy, i.e., the product in the twisted Hopf algebra  ${}_{\sigma}E(n)_{\sigma^{-1}}$  coincides with the product in  $E(n)$ . In this case the map  $\theta_{\sigma}$  is again a Hopf algebra map  $E(n)^{op} \rightarrow E(n)^*$  and Proposition 3.1 states that  $Cl(1, 0, \Lambda)^{op}$  is  $(E(n), r_A)$ -Azumaya if and only if  $\det(A - \Lambda - \Lambda^t) \neq 0$ . In particular, when  $\sigma$  is trivial we have:

**Corollary 3.2** *The comodule algebra  $E(n)^{op}$  is  $(E(n), r_A)$ -Azumaya if and only if  $\det(A) \neq 0$ .*

In the computation of the Brauer groups of  $H_4$ ,  $E(n)$  and  $H_{\nu}$  a special role is played by those universal  $r$ -forms that are non-zero only on the group algebra of the grouplike elements. For  $E(n)$  this is  $r_0$ . In this case we have:

**Corollary 3.3** *The comodule algebra  $Cl(1, 0, \Lambda)$  is  $(E(n), r_0)$ -Azumaya if and only if  $\det(\Lambda + \Lambda^t) \neq 0$ .*

It was shown in [9] that  $BM(\mathbf{k}, E(n), R_0)$  is isomorphic to the direct product of the Brauer-Wall group  $BW(\mathbf{k})$  of the field  $\mathbf{k}$  and the group  $Sym_n(\mathbf{k})$  of  $n \times n$  symmetric matrices with coefficients in  $\mathbf{k}$ , (represented by special cocycles cohomologous to those in [19]). On the other hand, the map  $\phi: E(n) \rightarrow E(n)^*$  with  $\phi(c) = 1^* - c^*$  and  $\phi(x_j) = x_j^* + (cx_j)^*$  for  $j = 1, \dots, n$

defines a Hopf algebra isomorphism. Therefore the pull-back along  $\phi$  yields an isomorphism

$$BM(\mathbf{k}, E(n)^*, r_0) = BM(\mathbf{k}, E(n)^*, (\phi \otimes \phi)(R_0)) \cong BM(\mathbf{k}, E(n), R_0).$$

Since  $BC(\mathbf{k}, E(n), r_0) \cong BM(\mathbf{k}, E(n)^*, r_0)$ , we may identify  $BC(\mathbf{k}, E(n), r_0)$  with  $BM(\mathbf{k}, E(n), R_0)$ . The class of  $Cl(\alpha, \gamma, \Lambda)^{op}$  in  $BC(\mathbf{k}, E(n), r_0)$  described above corresponds to the class of the algebra  $Cl(\alpha, \gamma, \Lambda)$  with action:

$$\begin{aligned} c \rightharpoonup u &= u\langle\phi(c), c\rangle = -u; & c \rightharpoonup v_i &= v_i\langle\phi(c), c\rangle + 1\langle\phi(c), x_i\rangle = -v_i; \\ x_i \rightharpoonup u &= u\langle\phi(x_i), c\rangle = 0; & x_j \rightharpoonup v_i &= v_i\langle\phi(x_j), c\rangle + 1\langle\phi(x_j), x_i\rangle = \delta_{ij}. \end{aligned}$$

We end this section describing the decomposition of the class represented by  $C(\alpha, \gamma, \Lambda)$  as a product of an element in  $BW(\mathbf{k})$  and an element in  $Sym_n(\mathbf{k})$ . We recall that the product  $\#$  corresponding to  $R_0$  and  $r_0$  is just the  $\mathbb{Z}_2$ -graded tensor product, where the grading is induced by the eigenspace decomposition with respect to the action of  $c$ .

Let us observe that, taking  $z_i = v_i - \frac{\gamma_i}{2\alpha}u$ , the algebra  $Cl(\alpha, \gamma, \Lambda)$  is isomorphic, as an  $E(n)$ -module algebra, to the algebra with generators  $u$  and  $z_1, \dots, z_n$ , relations:

$$\begin{aligned} uz_i + z_i u &= 0; & z_i z_j + z_j z_i &= 2\left(\frac{1}{4\alpha}\right)(2\alpha(\Lambda_{ij} + \Lambda_{ji}) - \Gamma_{ij}); \\ u^2 &= \alpha; & z_i^2 &= \left(\frac{1}{4\alpha}\right)(4\alpha\Lambda_{ii} - \Gamma_{ij}) \end{aligned}$$

and with action:

$$\begin{aligned} c \rightharpoonup u &= -u; & c \rightharpoonup z_i &= -z_i; \\ x_i \rightharpoonup u &= 0; & x_j \rightharpoonup z_i &= \delta_{ij}. \end{aligned}$$

Hence, we may always reduce to the case that the cleft extension is associated to a cocycle  $\omega$  with  $\omega(x_i \otimes x_j) = l_{ij}$  with  $L$  a symmetric matrix, and  $\omega(c \otimes x_j) = \omega(x_j \otimes c) = 0$  (see also [9, §2]). We shall denote such a module algebra by  $C(\alpha, L)$ . We observe that the map  $u \mapsto tu$  gives an isomorphism  $C(\alpha, L) \cong C(t^2\alpha, L)$ . The algebra  $C(\alpha, L)$  is isomorphic, as a  $\mathbb{Z}_2$ -graded algebra, to the Clifford algebra generated by the basis vectors  $u, z_1, \dots, z_n$  and with associated bilinear form corresponding to the matrix  $L = -\frac{1}{4\alpha}(2\alpha B + \Gamma)$ . Since  $\alpha$  is invertible, Proposition 3.1 in this case yields the well-known fact that a generalized Clifford algebra is  $\mathbb{Z}_2$ -graded central simple if and only if the associated bilinear form is non-degenerate.

Let us recall the decomposition of  $BM(\mathbf{k}, E(n), R_0)$  and the embedding of the subgroup  $Sym_n(\mathbf{k})$  described in [9]. The pull-back along the injection  $j$  of

$\mathbf{k}\mathbb{Z}_2$  into  $E(n)$  yields a surjective map  $j^*: BM(\mathbf{k}, E(n), R_0) \rightarrow BW(\mathbf{k})$ . The map  $j^*$  is split, the splitting map being induced by the pull-back  $p^*$  along the projection  $p: E(n) \rightarrow \mathbf{k}\mathbb{Z}_2$ . The Kernel of  $j^*$  is isomorphic to  $Sym_n(\mathbf{k})$  and representatives of its elements can be constructed as follows. To a symmetric matrix  $L$  one may associate a special 2-cocycle  $\omega$  such that

$$\begin{aligned}\omega(c \otimes c) &= 1; & \omega(x_i \otimes x_j) &= -l_{ij}; \\ \omega(c \otimes x_i) &= 0; & \omega(x_i \otimes c) &= 0.\end{aligned}$$

The left regular action of  ${}_{\omega}E(n)$  given by  $f_h(k) = \sum \omega(h_{(1)} \otimes k_{(1)})h_{(2)}k_{(2)}$  induces an inner action of  $E(n)$  on  $A^{\omega} = End({}_{\omega}E(n))$  by:  $h \cdot f = \sum f_{h_{(1)}} \circ f \circ f_{h_{(2)}}^{-1}$ . The module algebra  $A^{\omega}$  is  $(E(n), R_0)$ -Azumaya and it represents an element in the kernel of  $j^*$ . The subalgebra  $Ind(A^{\omega})$  generated by  $U = f_c$  and  $W_i = -f_{x_i}^{-1}$  for  $i = 1, \dots, n$  is a submodule algebra, and its relations are:

$$U^2 = 1, \quad W_i W_j + W_j W_i = 2l_{ij}, \quad U W_j + W_j U = 0.$$

The action on  $A^{\omega}$  is given by:

$$c \cdot f = U f U^{-1}; \quad x_j \cdot f = W_j(c \cdot f) - f W_j$$

for every  $f \in A^{\omega}$ . One shows that  $E(n)$  acts innerly on any representative  $A'$  of the class of  $A^{\omega}$ . If the action is realized by the map  $g: E(n) \rightarrow A'$  with  $g(c)$  and  $g^{-1}(x_j)$  skew-commuting, then the matrix  $L$  describing the relations among the  $g^{-1}(x_j)$ 's is an invariant of the class represented by  $A^{\omega}$  and it uniquely determines the class in the kernel of  $j^*$ .

For every quasitriangular Hopf algebra  $H$  and  $(H, R)$ -Azumaya algebra  $A$  let us denote by  $[A]$  the class in  $BM(\mathbf{k}, H, R)$  represented by  $A$ . Let us define, for any nonzero  $t \in \mathbf{k}$ , the algebra  $C(t)$  generated by  $x$  with relation  $x^2 = t$ ,  $c$ -action  $c \cdot x = -x$  and trivial action of the  $x_j$ 's. Being the representative of an element of  $BW(\mathbf{k})$ , the  $E(n)$ -module algebra  $C(t)$  is  $(E(n), R_0)$ -Azumaya. It is well-known that  $\overline{C(t)} = C(-t)$ . For every symmetric  $n \times n$  matrix  $L$  we denote by  $A(1, L)$  the  $E(n)$ -module algebra, representing  $p^* \circ j^*([C(1, L)])$ , isomorphic to  $C(1, L)$  as a  $\mathbf{k}\mathbb{Z}_2$ -module algebra and with trivial action of the  $x_j$ 's.

**Proposition 3.4** *Let  $\alpha$  be a nonzero element in  $\mathbf{k}$  and let  $L$  be an  $n \times n$  invertible, symmetric matrix with entries in  $\mathbf{k}$ . With notation as above*

$$[C(\alpha, L)] = [C(-1) \# C(\alpha) \# A(1, L)][A^{\omega}]$$

*with  $\omega(x_i \otimes x_j) = ((4L)^{-1})_{ij}$ .*

**Proof:** Let us first assume that  $\alpha$  is a square, so that  $[C(\alpha, L)] = [C(1, L)]$ . The class  $[C(1, L)] [\overline{A(1, L)}] = [C(1, L) \# \overline{A(1, L)}]$  belongs to the Kernel of  $j^*$ . We compute its matrix invariant.

Let us denote the generators of  $\overline{A(1, L)}$  by  $U$  and  $Z_1, \dots, Z_n$ . The relations in  $\overline{A(1, L)}$  are:

$$\begin{aligned} UZ_i + Z_iU &= 0; & Z_iZ_j + Z_jZ_i &= -2l_{ij}; \\ U^2 &= -1; & Z_i^2 &= -l_{ii} \end{aligned}$$

and the action is determined by

$$\begin{aligned} c \rightarrow U &= -U; & c \rightarrow Z_i &= -Z_i; \\ x_i \rightarrow U &= 0; & x_j \rightarrow Z_i &= 0. \end{aligned}$$

In the product  $C(1, L) \# \overline{A(1, L)}$  the elements  $U, Z_j$  for  $j = 1, \dots, n$  skew-commute with  $u$  and  $z_j$  for  $j = 1, \dots, n$ . Let us introduce the elements:

$$w_j = \sum ((-2L)^{-1})_{jk} z_k \in C(1, L).$$

It is not hard to see that  $x_j \rightarrow b = w_j(c \rightarrow b) - bw_j$  for  $b = u, w_j$  for  $j = 1, \dots, n$ . Since this equality extends to products, it holds for every  $b \in C(1, L)$ . Besides, for every  $a \# b \in C(1, L) \# \overline{A(1, L)}$  we have:

$$\begin{aligned} x_j \rightarrow (a \# b) &= (x_j \rightarrow a) \# (c \rightarrow b) + a \# (x_j \rightarrow b) \\ &= (x_j \rightarrow a) \# (c \rightarrow b) \\ &= w_j(c \rightarrow a) \# (c \rightarrow b) - aw_j \# c \rightarrow b \\ &= (w_j \# 1)(c \rightarrow (a \# b)) - (a \# b)(w_j \# 1). \end{aligned}$$

Since  $[C(1, L) \# \overline{A(1, L)}] \in \text{Ker}(j^*)$  the action of  $c$  on this product is strongly inner, i.e., there exists an element  $Y \in C(1, L) \# \overline{A(1, L)}$  with  $Y^2 = 1$  and such that  $c \rightarrow (a \# b) = Y(a \# b)Y^{-1}$ . In particular,  $Y(w_j \# 1) + (w_j \# 1)Y = 0$ . Therefore, the relations among the  $(w_j \# 1)$ 's for  $j = 1, \dots, n$  will give the sought invariant matrix. This is easily computed and we have:

$$(w_i \# 1)(w_j \# 1) + (w_j \# 1)(w_i \# 1) = 2((4L)^{-1})_{ij}.$$

Hence,

$$[C(t^2, L)] = [A(1, L)][A^\omega] \tag{3.3}$$

with  $\omega(x_i \otimes x_j) = ((4L)^{-1})_{ij}$ .

Let us assume now that  $\alpha$  is not a square. As  $E(n)$ -module algebras  $C(1) \# C(\alpha, L) \cong C(\alpha) \# C(1, L)$ . By formula (3.3) we have the statement.  $\square$

The Hopf algebra corresponding to  $n = 1$  is just Sweedler's Hopf algebra  $H_4$ . Proposition 3.1, together with self-duality of  $H_4$ , states that the module algebras  $A\langle \frac{\alpha, \beta, \gamma}{\mathbf{k}} \rangle$  in [22] are  $(H_4, R_t)$ -Azumaya if and only if  $2\alpha(t-2\beta)+\gamma^2 \neq 0$ . If  $\gamma = 0$ , we recover [23, Proposition 3.1]. When  $t = 0$ , we recover the result in [22] that the algebra  $A\langle \frac{\alpha, \beta, \gamma}{\mathbf{k}} \rangle$  is  $(H_4, R_0)$ -Azumaya if and only if  $-4\alpha\beta + \gamma^2 \neq 0$ . Up to a slight change in notation, Proposition 3.3 provides a bridge between the construction of the map  $(\mathbf{k}, +) \rightarrow BM(\mathbf{k}, H_4, R_0)$  in [22] and the construction of the map  $Sym_n(\mathbf{k}) \rightarrow BM(\mathbf{k}, E(n), R_0)$  in [9] for  $n = 1$ .

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## References

- [1] Auslander, M.; Goldman, O. The Brauer group of a commutative ring. Trans. Amer. Math. Soc. **1960**, *97*, 367–409.
- [2] Beattie, M.; Dăscălescu, S.; Grünefelder, L. Constructing pointed Hopf algebras by Ore extensions. J. Algebra **2000**, *225*, 743–770.
- [3] Blattner, R.J.; Cohen, M.; Montgomery, S. Crossed products and Galois extensions of Hopf algebras, Pacific J. Math **1989**, *137*, 37–54.
- [4] Caenepeel, S. Brauer Groups, Hopf Algebras and Galois Theory, Kluwer Academic Publishers (1998).
- [5] Caenepeel, S.; Van Oystaeyen, F.; Zhang, Y.H. Quantum Yang-Baxter module algebras. *K-theory*, **1994**, *8*, 231–255.
- [6] Carnovale, G. Some isomorphisms for the Brauer groups of a Hopf algebra. Comm. Algebra **2001**, *29* (11), 5291–5305.

- [7] Carnovale, G.; Cuadra, J. The Brauer group of some quasitriangular Hopf algebras. *Journal of Algebra* **2003**, *259*, 512–532.
- [8] Carnovale, G.; Cuadra, J. The Brauer group  $BM(k, D_n, R_z)$  of the dihedral group. *Glasgow Math. Journal* **2004**, *46*, 239–257.
- [9] Carnovale, G.; Cuadra, J. Cocycle twisting of  $E(n)$ -module algebras and applications to the Brauer group, arXiv:math.RT/0403444 v2.
- [10] Doi, Y.; Takeuchi, M. Cleft comodule algebras for a bialgebra. *Comm. Algebra* **1986**, *14*, 801–818.
- [11] Grothendieck, A. Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses, **1968** In: *Dix Exposés sur la Cohomologie des Schémas* 46–66 North-Holland, Amsterdam; Masson, Paris, 46–66.
- [12] Heyneman, R.G.; Radford, D.E. Reflexivity and coalgebras of finite type. *J. Algebra* **1974**, *28*, 215–246.
- [13] Long, F. W. A generalization of the Brauer group of graded Algebras. *Proc. London Math. Soc.* **1974**, *29*, 237–256.
- [14] Long, F. W. The Brauer group of dimodule algebras. *J. Algebra* **1974**, *31*, 559–601.
- [15] Majid, S. *Foundations of Quantum Group Theory*, Cambridge University Press (1995).
- [16] Montgomery, S. *Hopf Algebras and Their Actions on Rings*, CBMS 82, AMS (1993).
- [17] Pareigis, B. Non-additive ring and module theory. IV. The Brauer group of a symmetric monoidal category. In: *Brauer groups* (Proc. Conf., Northwestern Univ., Evanston, Ill., 1975), **1976**, *Lecture Notes in Math.* Vol. 549, 112–133.

- [18] Panaite, F.; Van Oystaeyen, F. Quasitriangular structures for some pointed Hopf algebras of dimension  $2^n$ . *Comm. Algebra* **1999**, *27* (10), 4929-4942.
- [19] Panaite, F.; Van Oystaeyen, F. Clifford-type algebras as cleft extensions for some pointed Hopf algebras. *Comm. Algebra* **2000**, *28* (2), 585-600.
- [20] Taylor, J. A bigger Brauer group. *Pacific J. Math.* **1982**, *103* (1), 163-203.
- [21] Van Oystaeyen, F.; Zhang, Y.H. The Brauer group of a braided monoidal category. *J. Algebra* **1998**, *202*, 96-128.
- [22] Van Oystaeyen, F.; Zhang, Y. The Brauer group of Sweedler's Hopf algebra  $H_4$ . *Proc. Amer. Math. Soc.* **2001** *129*, 371-380.
- [23] Van Oystaeyen, F.; Zhang, Y.; Computing subgroups of the Brauer group of  $H_4$ . *Comm. Algebra* **2002**, *30*(10), 4699-4709.
- [24] Wall, C.T.C.; Graded Brauer groups. *J. Reine Angew. Math.* **1964**, *213*, 187-199.